General asymptotic solutions of the Einstein equations and phase transitions in quantum gravity

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Abstract

We discuss generic properties of classical and quantum theories of gravity with a scalar field which are revealed at the vicinity of the cosmological singularity. When the potential of the scalar field is exponential and unbounded from below, the general solution of the Einstein equations has quasi-isotropic asymptotics near the singularity instead of the usual anisotropic Belinskii - Khalatnikov - Lifshitz (BKL) asymptotics. Depending on the strength of scalar field potential, there exist two phases of quantum gravity with scalar field: one with essentially anisotropic behavior of field correlation functions near the cosmological singularity, and another with quasi-isotropic behavior. The "phase transition" between the two phases is interpreted as the condensation of gravitons.

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One pessimistic quotation from the golden era of finding exact solutions of the Einstein equations which reflected the relations between particle theorists and experts in GR belongs to Richard Feynman. Taking part in the International Conference on Relativistic Theories of Gravitation at Warsaw, he was writing to his wife [1]: "I am not getting anything out of the meeting. I am learning nothing. ... I get into arguments outside the formal sessions (say, at launch) whenever anyone asks me a question or starts to tell me about his "work". The "work" is always: (1) completely un-understandable, (2) vague and indefinite, (3) something correct that is obvious and self-evident but worked out by a long and difficult analysis, and presented as an important discovery, or (4) a claim based on the stupidity of the author that some obvious and correct fact, accepted and checked for years, is in fact false ... (5) an attempt to do something probably impossible but certainly of no utility which, it is finally revealed in the end, fails ... or (6) just plan wrong ... Remind me not to come to any more gravity conferences!" Certainly, I am well aware of that the work presented in this essay could belong to the class (3) or (5) in the Feynman's classification (hopefully, not to the class (6)!), but I will follow Feynman's own words [1]: "We all do it for the fun of it" trying to find my fun in identifying some links which connect the part of the common lore on general relativity named "Exact solutions of the Einstein equations" to the problem of the GR quantization.

Of course, Feynman's interest was in the quantization of GR by applying the path integral approach working so well in QED. Solutions of the Einstein equations define saddle points of the action[2] $S = S_{\text{gravity}} + S_{\text{matter}}$ of the quantum gravity with matter. However, the contributions of these saddle points into the partition function of the theory and fluctuations near them

$$Z = \int \frac{Dg_{ik}}{Df} D\phi_{\text{matter}} \exp\left(-\frac{i}{\hbar}\left(S_{\text{gravity}} + S_{\text{matter}}\right)\right) \tag{1}$$

typically have *zero measure*. In other words, the probability for an almost any exact solution to describe the observable features of the Universe or some parts of it, to appear somehow from the quantum foam realized near the singularity is infinitely small, and the Feynman's anger is absolutely understandable.

Well, almost absolutely... Of course, there are several classes of solutions which will be important for the quantum part of the story, too, and one can without much thinking immediately identify some:

1. *Attractors*: among them are Minkowski spacetime, de Sitter (at least in the sense of eternal inflation [2]) and anti de Sitter spacetimes (a set of AdS domains is mostly probably the global attractor of GR realized as low-energy approximation of string theory [3]); black holes (Schwarzschild, Kerr, Reissner-Nordström, Kerr-Newman solutions), etc.

[2]From now on, by the quantum theory of gravity we mean effective QFT of spin 2 fields [4] (plus matter fields) — the one which particles with energies $E \ll M_P$ test. In this limit, the effects of the non-renormalizability may be neglected. Although we discuss below the situation which is realized near the cosmological singularity, we limit the discussion to time scales $t \gg t_P$.

2. *General solutions of the Einstein equations.* As usual [5], a solution of the Einstein equations is regarded as general if it contains sufficient number of arbitrary functions of coordinates. In the case of Ricci-flat spacetimes, this number is 4, and is equal to 8 in the presence of hydrodynamic matter.

While any non-attractor type solution of the Einstein equations defines the saddle point for the path integral (1) which does have a vanishing contribution into the overall partition function, eventually it well settle down towards an attractor solution due to the effect of classical perturbations and/or quantum fluctuations. The contribution of attractor type saddle points into the partition function (1) is therefore significant. However, the key word here is "eventually". For any non-attractor solution it takes a time $t_{\rm coll}$ before the solution reaches its attractor asymptotics.

Let us construct some initial state $|\Psi(t=t_i)\rangle$ of quantum matter fields in a curved spacetime and gravitons. The amplitude $\langle\Psi(t_f)|\Psi(t_i)\rangle$ is then defined by the path integral (1) calculated on the closed Schwinger-Keldysh contour from $t=t_i$ to $t=t_f$ and back. Then, if $t_f \ll t_{\rm coll}$, the corresponding attractor saddle point does not give any noticeable contribution into the amplitude.[3] If it is necessary to know the evolution of the quantum state $|\Psi(t)\rangle$ at time scales $t \ll t_{\rm coll}$, we are forced to pay much more attention to the type of saddle corresponding to general solutions of the Einstein equations.

Certainly, the Einstein equations are hard to solve, and it is possible to find something like their general solution only in physically simplified situations. As was first shown by Belinskii, Khalatnikov and Lifshitz [6], *asymptotically,* the general solutions of the Einstein equations near the cosmological singularity have the very same form for an almost arbitrary choice of the matter content. This asymptotics in the synchronous frame[4] is given by Kasner-like solution

$$ds^2 = dt^2 - \gamma_{\alpha\beta}(t,\mathbf{x})dx^\alpha dx^\beta, \tag{2}$$

$$\gamma_{\alpha\beta}(t,x) = t^{2p_1}\mathbf{l}_\alpha\mathbf{l}_\beta + t^{2p_2}\mathbf{m}_\alpha\mathbf{m}_\beta + t^{2p_3}\mathbf{n}_\alpha\mathbf{n}_\beta. \tag{3}$$

Both Kasner exponents p_1, p_2, p_3 and Kasner axis vectors $\mathbf{l}_\alpha$, $\mathbf{m}_\alpha$ and $\mathbf{n}_\alpha$ are arbitrary functions of space coordinates. The Einstein equations provide two constraints on the Kasner exponents

$$p_1 + p_2 + p_3 = 1, \tag{4}$$

and

$$p_1^2 + p_2^2 + p_3^2 = 1, \tag{5}$$

as well as three other constraints on arbitrary functions of space coordinates present in (3). Taking into account that the choice of synchronous gauge

$$g_{00} = 1,\ g_{0\alpha} = 0 \tag{6}$$

[3] Of course, the time scale $t_{\rm coll}$ itself is a functional of the initial state $|\Psi(t=t_i)\rangle$.

[4] Often, it is impossible to choose the globally synchronous frame of reference due to the limitations set by the casuality. However, everywhere in the text we discuss the physics in a given casual patch.

leaves the freedom to make three-dimensional space coordinate transformations, one can easily see that the total number of arbitrary coordinate functions in the Kasner-like solution (2),(3) is equal to 4 as it should be expected for a general solution of Einstein equations corresponding to an empty spacetime.

In the presence of the hydrodynamic matter Kasner solution (2),(3) describes asymptotic behavior of metrics near the singularity, [5] since components of energy-momentum tensor T_{ik} grow slower at $t \to 0$ then the components of the Ricci tensor.[6] Higher order corrections to the Kasner solution (2),(3), i.e., higher order terms in the expansion of $\gamma_{\alpha\beta}(t,x)$ over powers of t play the role of perturbations which give rise to the time dependence of Kasner exponents p_i as well as Kasner axis vectors $\mathbf{l}_\alpha$, $\mathbf{m}_\alpha$ and $\mathbf{n}_\alpha$ and to well-known BKL chaotic behavior. Therefore, the BKL solution is simultaneously a *universal attractor* for all solutions of the Einstein equations possessing a spacelike singularity. It means that *no other saddle points contribute* into the amplitude (1) in the vicinity of the cosmological singularity.

In this essay, it will be first of all shown that in the presence of a scalar field with potential $V(\phi)$ which is exponential and unbounded from below, the general asymptotic solution of the Einstein equations is different from the BKL solution and is quasi-isotropic [8] (while the BKL solution is essentially anisotropic). In particular, we will choose potential of the form[7]

$$V(\phi) = -|V_0| \mathrm{ch}\,(\lambda\phi)\,. \tag{7}$$

Scalar field potentials of this form appear in problems related to gauged supergravity models [10] and the ekpyrotic scenario [11]. The cosmological singularity realized in such theory is of the Anti de Sitter Big Crunch type. The physics in its vicinity it is interesting by itself and even more so since this type of singularity seems to be realized quite often on the string theory landscape [3].

As in the case discussed in [6], it is convenient to perform all calculations in the synchronous frame of reference where $g_{00} = 1$, $g_{0\alpha} = 0$, $g_{\alpha\beta} = -\gamma_{\alpha\beta}$, $\alpha, \beta = 1\ldots 3$, i.e., the spacetime interval has the form

$$ds^2 = dt^2 - \gamma_{\alpha\beta}(t,x)dx^\alpha dx^\beta. \tag{8}$$

Near the hypersurface $t = 0$ which corresponds to the singularity, the spatial metric components behave as

$$\gamma_{\alpha\beta}(t,\mathbf{x}) = a_{\alpha\beta}(\mathbf{x})t^{2q} + c_{\alpha\beta}(\mathbf{x})t^d + b_{\alpha\beta}(x)t^n + \sum_{i,j} d^{(i,j)}_{\alpha\beta}(x)t^{f_{ij}}\,. \tag{9}$$

With the same precision, one has in the vicinity of singularity

$$\phi(t,\mathbf{x}) = \psi(x) + \phi_0(\mathbf{x})\log(t) + \phi_1(x)t^{f_1} + \phi_2(x)t^{f_2} + \cdots, \tag{10}$$

[5] Which corresponds everywhere below to the spacelike hypersurface $t = 0$.

[6] If there is a scalar field in the matter content [7], BKL solution (3) remains general solution of the Einstein equations with changed Kasner constraints (4),(5).

[7] The quasi-isotropic solution for such potentials was first found at the background level in [9], where it was also shown that it is the attractor. The goal we pursue in this essay is to prove that the quasi-isotropic solution is also general and to understand how its instability develops with the change of the form of the potential.

with dots corresponding to higher order terms of $\phi(t,\mathbf{x})$ expansion in powers of t. From the Einstein equations one finds[8] that the leading exponents in the expansions (9) and (10) are defined by the expressions[9]

$$q \;=\; \frac{16\pi}{M_P^2\lambda^2},\ n=2,\ d=1-q, \tag{11}$$

$$\psi(x)=\mathrm{Const},\ \phi_0(x)=\frac{2}{\lambda},\ f_1=1-3q,\ f_2=2-q, \tag{12}$$

$$c^\alpha_\alpha(x)=2\lambda\phi_1(x),\ c^\beta_{\alpha;\beta}(x)=\frac{1-2q}{1-3q}\frac{16\pi}{M_P^2}\phi_0\phi_{1,\alpha}(x), \tag{13}$$

$$\tilde{P}^\beta_\alpha(x)+(1-q)(qb^\gamma_\gamma(x)\delta^\beta_\alpha+(1+q)b^\beta_\alpha(x))=\frac{4\pi V_0}{M_P^2}e^{-\psi}\lambda\phi_2(x), \tag{14}$$

$$-(1-q)b^\alpha_\alpha(x)=\frac{32\pi}{M_P^2}(1-q)\phi_0\phi_2(x)-\frac{4\pi}{M_P^2}V_0e^{-\psi}\lambda\phi_2(x), \tag{15}$$

where $\tilde{P}^\beta_\alpha(x)$ is the 3-dimensional Ricci tensor constructed from components of the tensor $a^\beta_\alpha(x)$ as components of metric tensor. Higher order terms in the expansions (9) and (10) can be selfconsistently calculated by using the Einstein equations and the orthogonality condition

$$\gamma^\beta_\alpha\gamma^\lambda_\beta=\delta^\lambda_\alpha. \tag{16}$$

One can immediately find from Eq. (16) that the higher order exponents in the metric (9) are defined by

$$f_{ij}=i+2j-(3i+2j-2)q, \tag{17}$$

where $i,j\in\mathbb{N}$. The n term in the metric expansion corresponds to $i=0,\ j=1$ and d term — to $i=1,\ j=0$. It is easy to see that there is no other exponents in the expansion (9).

Let us examine the formulae (11)-(15) more closely and calculate the number of arbitrary functions present in this solution. First of all, one can immediately see that the tensor $a^\beta_\alpha(x)$ is not constrained by the Einstein equations. It has 6 components, and 3 of them can be made to be equal to 0 by a three-dimensional coordinate transformation (the remnant gauge freedom of the synchronous gauge (6)). Since this tensor is used for lowering and rising the indices and represents the leading term in the expansion (9), we will identify the term $a_{\alpha\beta}t^{2q}$ as a *background* contribution to $\gamma_{\alpha\beta}(t,x)$. Furthermore, we see from Eqs. (14),(15) that $b_{\alpha\beta}$ can be reconstructed from the known tensor $a_{\alpha\beta}$.

The tensor $c_{\alpha\beta}$ contains three more arbitrary functions of coordinates. Indeed, it can be represented in the form

$$c^\beta_\alpha(x)=\frac{1}{3}c^\gamma_\gamma\delta^\beta_\alpha+Y^\beta_{;\alpha}+Y^{;\beta}_\alpha-\frac{2}{3}Y^\gamma_\gamma\delta^\beta_\alpha+c^{(\mathrm{TT})\beta}_\alpha. \tag{18}$$

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[8] Due to the limitations of space we are unable to present the full derivation of the solution here. It will be given in the forthcoming publication [12].

[9] The indices of all matrices are lowered and raised by the tensor $a_{\alpha\beta}$, for example, $b^\beta_\alpha=a^{\beta\gamma}b_{\gamma\alpha}$.

From Eq. (13) one can see that its trace part defines the value of $\phi_2(x)$ contributing to Eq. (10) and therefore provides one arbitrary function. Then, three components of the vector contribution $Y_\alpha(x)$ are fixed, and transverse traceless part $c_\alpha^{(\mathrm{TT})\beta}(x)$ provides remaining two arbitrary functions. We also note that the $c_{\alpha\beta}$ term can be regarded as the leading term *perturbation* to the background contribution into $\gamma_{\alpha\beta}$. In particular, it contains the contribution of scalar perturbations (related to the trace of the tensor c_α^β) and tensor perturbations or gravitons (related to the transverse traceless part of the tensor c_α^β).

The total number of arbitrary functions in the solution (9),(10) is therefore 6, as one may expect for the general solution of the Einstein equations with a scalar field. By analysis similar to [6], one may show [9, 12] that the contributions of other matter fields into the overall energy-momentum tensor grow slower at $t \to 0$ than the contribution of the scalar field. We conclude that the solution (9), (10) is the general asymptotic solution of Einstein equations (with arbitrary matter content) near the cosmological singularity. Similarly to the BKL solution, the quasi-isotropic solution is *the universal attractor* for all solutions of the Einstein equations with scalar field having the potential (7) and arbitrary additional matter content which possess the time-like singularity. Again, under considered conditions, no other saddle points contribute into the amplitude (1) in the vicinity of the Big Crunch singularity.

It is instructive to understand how exactly the transition from the quasi-isotropic regime (9),(10) near the singularity to the BKL anisotropic regime (3) happens. This transition can be achieved by changing the value of λwhile keeping V_0 fixed (or vise versa).

By construction, $2q < d = 1 - q$, i.e., the exponent $a_{\alpha\beta}t^{2q}$ in the expansion (9) is leading. With the increase of q, the value of d decreases and when q reaches the critical value $q_c = 1/3$, the contributions $a_{\alpha\beta}(x)t^{2q}$ and $c_{\alpha\beta}(x)t^d$ into the expansion of the metric (9) become of the same order. Similarly, one can check that the values of higher order exponents (17) decrease with the increase of q. In particular, all exponents with different i's and similar j's become of the same order of magnitude at $q_c = 1/3$. At $q > q_c = 1/3$ the general asymptotic solution of the Einstein equations near the singularity is given by Eq. (3) instead of Eq. (9).

In fact, what we have just found is relevant for the quantum part of the story, too, and in a sense is analogous to the *spontaneous symmetry breaking* phenomenon in QFTs. Indeed, let us take the theory with a scalar field

$$\mathcal{L} = \frac{1}{2}\partial_i\Phi\partial^i\Phi - \frac{\lambda}{4}(\Phi^2 - v)^2, \tag{19}$$

set $\Phi(x,0) = 0$ as an initial condition and continuously change the value of the parameter v. At $v > 0$ the solution $\Phi(t,x) = 0$ of the classical equations of motion is perturbatively stable and corresponds to the true vacuum of the theory at the quantum level. At $\nu < 0$ the same solution becomes classically unstable, and $\Phi(t,x)$ reaches the "true" vacuum value $\Phi = \pm\sqrt{v}$ during the time $t \sim \frac{1}{\lambda\sqrt{v}}\log\frac{1}{\lambda}$ (with the VEV of the operator $\hat{\Phi}$ having similar behavior at the quantum level). Similar situation is realized in our case.

At $q < q_c = 1/3$ the quasi-isotropic solution (9),(10) is the general solution of the Einstein equations; it is perturbatively stable by construction (without any limitations on the weakness of the perturbations). At $q > q_c$ the quasi-isotropic solution becomes perturbatively unstable (perturbations defined by $c_{\alpha\beta}$ and higher order terms grow faster than the background term $a_{\alpha\beta}$ at $t \to 0$).

Vise versa, at $q > q_c = 1/3$ the BKL anisotropic solution if the Einstein equations is general in the vicinity of the cosmological singularity. It is stable by construction with respect to arbitrary perturbations and the stability is lost at $q < q_c$.

This analysis remains valid for the quantum situation[10] since the canonical phase space is in one-to-one correspondence with the space of solutions of classical field equations [13], and both quasi-isotropic and BKL solutions are (a) general and (b) universal attractors for other solutions of the Einstein equations in the vicinity of the time-like singularity.

The transition from the regime realized at $q < 1/3$ to the regime $q > 1/3$ probably corresponds in the quantum level to the condensation of gravitational perturbations. Indeed, one can interpret the higher order contributions in the expansion (9) as terms corresponding to the *interaction* between gravitational degrees of freedom as well as higher order nonlinearities in the background. Our conclusion is based on the fact that at $q = q_c$ the spectrum of the exponents in the expansion (9) becomes infinitely dense. It is also possible to show that the point of the "phase transition" $q_c = 1/3$ corresponds at the classical level to the situation when the choice of globally synchronous frame of reference is impossible near the singularity [12].

Let us summarize what have been found in the present essay. We have shown that in the presence of the scalar field with exponential potential unbounded from below, the general asymptotic solution of the Einstein equations near the cosmological singularity has quasi-isotropic behavior instead of anisotropic found by [6]. We have argued that at the quantum level there should exist a phase transition between the quasi-isotropic and anisotropic phases, governed by the strength of the scalar field potential and interpreted this phase transition as the condensation of gravitational perturbations.

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[10] One important comment regarding the quantization should be made. The quantum theory of the scalar field with the potential (7) is tachyonically unstable and has neither well-defined asymptotic $|\text{out}\rangle$ states, nor $\langle\text{out}|\text{in}\rangle$ S-matrix. However, the Schwinger-Keldysh $\langle\text{in}|\text{in}\rangle$ S-matrix is defined, and it is possible to make sense of the corresponding time-dependent theory [12].

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